FINAL-SEMESTER EXAMINATION
B. MATH III YEAR, II SEMESTER April 2017
ANALYSIS IV.

1. Show that the set of irrational numbers with ususal metric is a separable metric space.

Solution: This was done in a previous midsem paper. Get the solution from the separability of $\mathbb{R}$.

2. Let X be a complete metric space and A be a totally bounded subset of X. Show that $\overline{A}$ is compact.

Solution: This is a standard theorem: A is compact iff it is closed and bounded. If A is totally bounded, so is its closure. Now imitate the proof of the mentioned result.

3. Let X be a separable metric space and U_α be a family of open sets covering X. Give a detailed proof that there is a countable subcover for of this family.

Solution: Fix a countable dense set (x_j) is X. Consider open balls of radius $1/m$ (m - natural number) around each x_j.

4. Consider $\ell^2 = \{(x_n) : \sum_n |a_n|^2 < \infty$ with the usual metric and B be the set $\{(x)_n \in \ell^2 : \sum_n |a_n|^2 \leq 1\}$. Show that B is not compact.

Solution: Get a sequence in B that has no convergence subsequence in the usual metric.

5. Let $f : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function and $f(\mathbb{R})$ is contained in $\{x \in \mathbb{R}^3 : \|x\| = 1\}$. Show that $f(t)$ is orthogonal to $f'(t)$ for all t.

Solutiion: $\|f(t)\| = 1$ implies $f_1(t)^2 + f_2(t)^2 + f_3(t)^2 = 1$ where $f = (f_1, f_2, f_3)$. Differentiating this we get $f_1(t)f_1'(t) + f_2(t)f_2'(t) + f_3(t)f_3'(t) = 0$ which gives othogonality.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuosly differentiable, and $f'(0)$ has a non-zero determinant. If $U = \{x \in \mathbb{R}^2 : \|f'(x) - f'(0)\| < \frac{1}{1\|f^prime(0)\|}\}$, then $f(U)$ is open.

: Solution: See Rudin. 7. Let $f : [1, \infty) \to \mathbb{R}$ be a Riemann integrable on $[1, a]$ for all $a > 1$. Let $\int_1^\infty f(t)dt$ converges abosultely. Show that

$$\lim_{\alpha \to \infty} \int_1^\infty f(t)Sin(\alpha t)dt = 0.$$

Solution: See Baby Rudin and Royden.

8. Let g be a continous function of bounded variation with $g(1) = 1$. Derive the formula

$$\pi = 2 \lim_{\alpha \to \infty} \int_1^\infty f(t) Sin(\alpha t)/t dt.$$

See Royden.

9. Let φ_n be an orthonormal sequence on $[0, \pi]$. Let $\mathcal{R}$ denote the set of all Riemann integrable functions on $[0, \pi$. For $f \in \mathcal{R}$, let $a_n = \int_0^\pi f(t)\overline{\varphi_n(t)}$. Suppose for all $f \in \mathcal{R}$, $\int_0^\pi |f(t)^2 dt = \sum_n |a_n|^2$. If (a_n) and (b_n) are the Fourier coefficients of f and g show that $\sum_n a_n \overline{b_n}$ converges, equals to $< f, g >$.

10. Let ψ_n be an orthonormal sequence on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be Riemann integrable. If the Cesaro sum corresponding to the Fourier series converges uniformly, then

$$\int_a^b |f(t)|^2 = \sum_n |a_n|^2,$$

where $a_n = \int_a^b f(t)\overline{\psi_n(t)}$.

The solutions to above questions are standard, see Royden, Rudin or Simmons.